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Monochrome symmetric subsets of colored groups

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Abstract

In (Electron. J. Combin. 10 (2003); http://www.combinatorics.org/volume-10/Abstracts/v1oi1r28 html), the first author (Yuliya Gryshko) asked three questions. Is it true that every infinite group admitting a 2-coloring without infinite monochromatic symmetric subsets is either almost cyclic (i.e., have a finite index subgroup which is cyclic infinite) or countable locally finite? Does every infinite group G include a monochromatic symmetric subset of any cardinal $< |G|$ for any finite coloring? Does every uncountable group G such that $|B(G)| < |G|$ where $B(G) = \{x \in G : x^2 = 1\}$, admit a 2-coloring without monochromatic symmetric subsets of cardinality $|G|$? We answer the first question positively. Assuming the generalized continuum hypothesis (GCH), we give a positive answer to the second question in the abelian case. Finally, we build a counter-example for the third question and we give a necessary and sufficient condition for an infinite group G to admit 2-coloring without monochromatic symmetric subsets of cardinality $|G|$. This generalizes some results of Protasov on infinite abelian groups (Mat. Zametki 59 (1996) 468–471; Dopovidi NAN Ukrain 1 (1999) 54–57).

Keywords: Coloring; Monochromatic; Group; Symmetric

We will use a number of straightforward and mostly standard group theoretic notions which for the reader's convenience are recalled below.

Definitions. Let G be a group. A subset A of G is *symmetric* with respect to the element g of G if and only if $gA^{-1}g = A$. A subset A of G is *symmetric* if it is *symmetric* with respect to some element g of G.

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For any element g of G the *order* of g is the cardinal of the subgroup $< g >$ generated by G.

A group G is called *locally finite* if every finitely generated subgroup of G is finite.

A group G is called *almost cyclic* if G has a subgroup H of finite index which is isomorphic to $(\mathbb{Z}, +)$.

A group G is called a *torsion group* if every element of G is of finite order.

Let p be a prime number. A group G is called a *p-group* if the order of every element of G is a power of p.

Let x and y be two elements of G, we denote by $[x, y]$ the element $x^{-1}y^{-1}xy$.

Let G be a group, we denote by $[G, G]$ the derived subgroup of G, i.e., the subgroup generated by the set $\{[x, y] : x \in G,\ y \in G\}$.

Let G be a group, we denote by $B(G)$ the set $B(G) = \{x \in G : x^2 = 1\}$.

Theorem 1. *Let G be an infinite group. Let c be a color function from G to* $\{0, 1\}$ (*i.e., just a function assigning one of two colors*). *Suppose that G does not include any infinite symmetric monochromatic subset. Then G is either countable locally finite or almost cyclic.*

From Lemmas 1–5 *we assume the hypothesis of Theorem* 1.

Lemma 1. 1. *For every element a of G,* $\{x \in G : c(x) \neq c(axa)\}$ *is finite.* 2. *For every element a of* $[G, G]$, $\{x \in G : \text{either } c(x) \neq c(ax) \text{ or } c(x) \neq c(xa)\}$ *is finite.*

Proof. According to the assumptions of Theorem 1, the four following subsets of G:

$\{x \in G : c(x) = c(x^{-1}) = 0\}$, $\{x \in G : c(x) = c(x^{-1}) = 1\}$,

$\{x \in G : c(x^{-1}) = c(axa) = 0\}$, $\{x \in G : c(x^{-1}) = c(axa) = 1\}$ are finite. Since $\{x \in G : c(x) \neq c(axa)\}$ is obviously included in the union of these four subsets of G, it is finite. Thus we have proved (1).

Let g and h be two elements of G. According to (1), the three maps from G to G:

$x \to gxg$,

$x \to hxh$,

$x \to (hg)^{-1}x(hg)^{-1}$,

leave the color invariant except for finitely many x. Thus composition $x \to xghg^{-1}h^{-1}$ leaves the color invariant except for a finite number of x. By the same reasoning, left multiplication by a commutator leaves the color invariant except for a finite number of elements of G. It follows that both right and left multiplication by an element of $[G, G]$ leaves the color invariant except for a finite number of elements of G. Thus we have proved (2). □

Lemma 2. *Let a be any element of G such that the image of a by the canonical morphism* $G \to G/[G, G]$ *is of infinite order. Then for any element b of* $[G, G]$, *the set* $\{a^{-n}ba^n : n \in \mathbb{Z}\}$ *is finite.*

Proof. Assume the contrary. Denote by φ the quotient morphism from G to $G/[G, G]$. Then $\varphi(a)$ is of infinite order and there is an element b of $[G, G]$ such that the set $B = \{a^{-n}ba^n : n \in \mathbb{Z}\}$ is infinite. As $[G, G]$ is a normal subgroup of G, one has $B \subset [G, G]$. Thus $[G, G]$ is infinite. For every element x of $[G, G]$, we consider the subset $D(x)$ of G

defined as $D(x) = \{a^n x a^n : n \in \mathbb{Z}\}$. Let x and y be two elements of $[G, G]$, the intersection of the subsets $D(x)$ and $D(y)$ is not empty if and only if there are n and m in $\mathbb{Z}$ such that $a^n x a^n = a^m y a^m$. But if $a^n x a^n = a^m y a^m$, $\varphi(a)^{2n} = \varphi(a^n x a^n) = \varphi(a^m y a^m) = \varphi(a)^{2m}$. As $\varphi(a)$ is of infinite order, we can conclude that $n = m$ which implies that $x = y$. According to the above sentence and (1) of Lemma 1, the subsets $D(x)$ of G are monochromatic except for finitely many x of $[G, G]$. Thus except for finitely many n of $\mathbb{Z}$, the set $D(a^{-n} b a^n)$ is monochromatic. It follows that except for finitely many n of $\mathbb{Z}$, we have $c(a^{-2n} b) = c(a^{-n} b a^n) = c(b a^{2n})$.

Let I be the subset of $\mathbb{Z}$ defined as $I = \{n \in \mathbb{Z} : c(a^{-2n}) = c(a^{-2n} b) = c(a^{-n} b a^n) = c(b a^{2n}) = c(a^{2n})\}$. According to (2) of Lemma 1, I is a cofinite subset of $\mathbb{Z}$. Namely, the set $\{a^{2n}, a^{-2n} : n \in I\}$ is an infinite monochromatic subset of G that is symmetric with respect to 1. This contradicts the nonexistence of infinite symmetric monochromatic subsets of G. □

Lemma 3. (*Main lemma*). *Either $G/[G, G]$ is a torsion group or $[G, G]$ is finite.*

Proof. Assume the contrary. Let a be an element of G such that $\varphi(a)$ is of infinite order. For every element x of $[G, G]$, the subset $D(x)$ of G is defined as $D(x) = \{a^n x a^n : n \in \mathbb{Z}\}$. By the reasoning used in Lemma 2, the subsets $D(x)$ are monochromatic except for finitely many x. Since $[G, G]$ is infinite, there exists an element b of $[G, G]$ such that $D(b)$ is monochromatic. According to Lemma 2, there is a nonzero element m of $\mathbb{Z}$ such that a^m commutes with b. Let H the subgroup of G generated by a^{2m}. Then $bH \subset D(b)$. It follows that bH is an infinite monochromatic subset of G which is symmetric with respect to b; this is a contradiction. □

Lemma 4. *G is countable.*

Lemma 4 *was proved in* [3], *but for the convenience of the reader, we will give a proof of this statement.*

Proof. *Case* 1: $[G, G]$ *is infinite*: Let H be an infinite countable subgroup of $[G, G]$. For every x in H, the coset xH is an infinite symmetric subset of G with respect to x. Thus it is not monochromatic. We can conclude that for every element x of G, there exists h in H such that $c(x) \neq c(xh)$. It follows that G is the union of the finite sets (see Lemma 1) $E(h) = \{x \in G : c(x) \neq c(xh)\}$. Thus G is countable.

Case 2: $[G, G]$ *is finite*: According to (2) of Lemma 1, except for finitely many x, the cosets $x[G, G]$ are monochromatic. It follows that without loss of generality we can suppose that G is abelian (as we can replace G by $G/[G, G]$). Since there are no monochromatic symmetric infinite subsets of G, the set $B(G)$ is finite. Thus the subgroup G^2 of G, $G^2 = \{x^2 : x \in G\}$, is infinite. From (1) of Lemma 1 and the commutativity of G, we deduce that for every g in G^2, $\{x \in G : c(x) \neq c(xg)\}$ is finite. By taking an infinite countable subgroup H of G^2, we can do the same reasoning as in the first case and prove that G^2 is countable, which implies, since $B(G)$ is finite, that G is countable. □

Lemma 5. *Let H be a finitely generated subgroup of* $[G, G]$. *Either H is finite or H is almost cyclic and of finite index in G.*

This Lemma was proved in [3]. *For the convenience of the reader we will give a proof.*

Proof. Assume that H is infinite. Let X be a finite set of generators of H. According to König's lemma, there is a sequence $(u_n)_{n\in\mathbb{N}^*}$ of elements of $X \cup X^{-1}$ such that if for every element x of H, $l(x)$ is the *length* of x (the minimal number of elements of $X \cup X^{-1}$ needed to write x), $l(u_1 \ldots u_n) = n$. Let A be the following subset of H : $A = \{1, u_1, \ldots, (u_1 \ldots u_n) \ldots\}$.

The set $A \cup A^{-1}$ is infinitely symmetric with respect to 1, thus it is not monochromatic and contains infinitely many elements of each color. According to (2) of Lemma 1, for every element x of H, one has $c(xg) = c(gx) = c(g)$ except for finitely many g of G, thus $(xA) \cup (A^{-1}x)$ is not monochromatic. Let D be defined as follows:

$$D = \{x \in G : \exists v \in X \cup X^{-1},\ \text{either}\, c(xv) \neq c(x) \,\text{or}\, c(vx)) \neq c(x)\}.$$

The set D is finite. For any element x of H, the fact that $(xA) \cup (A^{-1}x)$ is not monochromatic implies that $((xA) \cup (A^{-1}x)) \cap D$ is not empty:

Let R be the following relation on G: sRt if and only if there is an element v of $X \cup X^{-1}$ such that $sv = t$. For any element x of H, $(xA) \cup (A^{-1}x)$ is obviously a connected graph for the relation R. Since $(xA) \cup (A^{-1}x)$ is not monochromatic, we can find two elements c and d of $(xA) \cup (A^{-1}x)$ with different colors. Then these two elements belong to D.

Thus $H = A(D \cap H) \cup (D \cap H)A^{-1}$. The set D is finite in view of (2) of Lemma 1. Since A contains, for each nonnegative integer n, exactly one element of length n, we can conclude that the growth of H is linear. But by the main theorem of Gromov [2], every group of polynomial growth has a nilpotent subgroup of finite index. By Proposition 2 p.76 of Tits [7], every nilpotent group of linear growth is almost cyclic. It follows that H is almost cyclic. Now let us prove that the index of H in G is finite. The connected components of the graph of R are obviously the cosets xH where x belongs to G. These cosets being infinite symmetric, they are not monochromatic. By the same reasoning, as in the above paragraph with the cosets replacing $(xA) \cup (A^{-1}x)$, we can deduce that $G = HD$. It follows from the finiteness of D that the index of H in G is finite. Thus we have proved Lemma 5. □

Proof of Theorem 1. *Case* 1: $G/[G, G]$ *contains an element of infinite order*: Let a be an element such that $\varphi(a)$ is of infinite order (with the notations of Lemma 2). Let $C(a)$ be the centralizer of a in G. As $[G, G]$ is finite (consequence of Lemma 3), $C(a)$ is a subgroup of G of finite index. This follows from the fact, easily verified, that b and c belong to the same right residue class of $C(a)$ if $[a, b]$ and $[a, c]$ are equal. According to Lemma 1, $\{x \in C(a) : c(x) \neq c(axa)\}$ is finite.

Thus $\{x \in C(a) : c(x) \neq c(xa^2)\}$ is finite. Let H be the cyclic subgroup of $C(a)$ generated by a^2. The cosets xH, x belonging to $C(a)$, are monochromatic except for finitely many of them. But they are infinite symmetric subsets of G. Thus there are only finitely many such cosets, which means that H is a subgroup of finite index of $C(a)$. As $C(a)$ is a subgroup of finite index of G, the index of H in G is finite. Then G is almost cyclic.

Case 2: $G/[G, G]$ *is a torsion group*: $G/[G, G]$ is an abelian torsion group. So, it is locally finite. Either $[G, G]$ is locally finite and G is locally finite countable (the countability of G is the content of Lemma 4) or $[G, G]$ includes an infinite finitely generated subgroup H. According to Lemma 5, H is almost cyclic and of finite index in G. In this case, G is almost cyclic too. □

Theorem 2. *Assume the generalized continuum hypothesis* (*GCH*). *Let G be an infinite abelian group. Let c be a color function from G into a finite set F. Then there are monochromatic symmetric subsets of G for any cardinality* $< |G|$.

Proof.

Step 1: We will first prove that if $G^2 = \{x^2 : x \in G\}$, then either G has a monochromatic symmetric subset of cardinality $|G|$ or $\left|G^2\right| = |G|$.

Assume the contrary. Note that G^2 is a subgroup of G and that $x \to x^2$ is a morphism of G onto G^2 whose kernel is $B(G)$. If $\left|G^2\right| < |G|$, then we obviously have

$|B(G)| = |G|$. Since F is finite, $B(G)$ includes a monochromatic subset A of cardinality

$|B(G)| = |G|$. But every subset of A is symmetric with respect to 1, to this is a contradiction.

Step 2: We can assume by the previous step that the groups G and G^2 have the same cardinal.

Let κ be a cardinal such that $\kappa < |G|$. Let $(\kappa + 1)$ be the successor of κ as ordinal.

According to a theorem of Erdős and Rado (cf.[1] (10) p. 95: (10)) is used with κ replacing λ and ρ in (10), κ^+ replacing κ, and 1 replacing τ) and *GCH*, there exists an element b of F and a $(\kappa + 1)$-sequence g_i of different elements of $G^2(i \in (\kappa + 1))$ such that if $i < j$, $c(g_j g_i^{-1}) = b$.

Thus the set $\{g_i g_0^{-1}, g_\kappa g_i^{-1} : i \in (\kappa + 1)\}$ is monochromatic of cardinality κ and symmetric with respect to every square root of $g_\kappa g_0^{-1}$ (it exists because the g_i are elements of G^2).

We have proved Theorem 2. □

Theorem 3. *For every infinite cardinality* κ, *there exists a group G with the three following properties*:

(1) $|G| = \kappa$,
(2) $|B(G)| < |G|$.
(3) *For every* 2*-coloring of G, there is a monochromatic symmetric subset of G of cardinality* $|G|$.

For κ *uncountable this provides the announced counter-example to a question of Gromov* [2].

Proof. Assume the contrary. Let K be a field of cardinality κ and of characteristic $\neq 2$.

We claim that we can take for G the special linear group $SL_2(K)$. One has $|G| = \kappa$ and $B(G) = \{ - Id, +Id\}$ where *Id* denotes the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus the first two conditions are satisfied. We claim that the third condition is satisfied. We argue by contradiction. Let c be a 2-coloring without monochromatic symmetric subsets of G of cardinality $|G|$ [4,5]. By the same reasoning as in Lemma 1 of Theorem 1, for every element a of $[G, G]$, $|\{x \in G : c(x) \neq c(xa)\}| < |G|$. For every infinite field K of characteristic $\neq 2$, every normal subgroup of $SL_2(K)$ which is not included in $\{ - Id, Id\}$ is $SL_2(K)$ (cf. [6, p. 226] the proof of Theorem 8.13 which shows in this fashion that $PSL_2(K)$ is simple). It follows that $SL_2(K) = [SL_2(K), SL_2(K)]$ and that $-Id \in [SL_2(K), SL_2(K)]$. Thus we have $|\{x \in G : c(x) \neq c(-x)\}| < |G|$.

Let $I(G)$ be the subset of G defined as $I(G) = \{x \in G : x^2 = -Id\}$. Let E be the subset of G defined as $E = \{x \in I(G), c(x) = c(-x)\}$. The set $I(G)$ contains obviously all the matrices of the form $\begin{pmatrix} a & a^2+1 \\ -1 & -a \end{pmatrix}$ with $a \in K$. It follows that $I(G)$ has the same cardinality as G. Then E has the same cardinality as G and any maximal monochromatic subset of E is symmetric with respect to 1. It follows that G includes a monochromatic symmetric subset of cardinality $|G|$. This is a contradiction. □

Theorem 4. *Let G be an infinite group. If either G is locally finite countable or G is uncountable, the two following statements are equivalent*:

(1) *There exists a 2-coloring c of G without monochromatic symmetric subsets of G of cardinality* $|G|$.
(2) *Let* $L(G)$ *be the subgroup of* $G \times G$, $L(G) = \{(x, y) \in G \times G, xy \in [G, G]\}$. *There is a chain C (for inclusion) of subgroups K of* $L(G)$ *each of cardinality* $< |G|$ *such that the union of the K is* $L(G)$ *and such that for every K of C*,

$$\left|\{x \in G : \exists (u, v) \in K\ x^2 = uv\}\right| < |G|$$

(*if* $|G|$ *is a regular cardinal this statement is obviously equivalent to*

$\forall a \in [G, G] \left|\{x \in G : x^2 = a\}\right| < |G|$).

Proof. (1) ⇒ (2): Suppose (1). By the same reasoning as in of Theorem 1, for every element a of $[G, G]$, $|\{x \in G : c(x) \neq c(xa)\}| < |G|$ and for every element b of G, $|\{x \in G : c(x) \neq c(bxb)\}| < |G|$.

Thus for every d, f of G such that $d, e \in [G, G]$,
$\left|\{x \in G : c(x) \neq c(d^{-1}xf)\}\right| < |G|$.

If G is locally finite countable, we take any chain of finite subgroups of $L(G)$ whose union is $L(G)$. If G is uncountable, for every infinite cardinal $\lambda < |G|$, let us consider the group

$V_\lambda = \{(g, h) \in L(G) : \left|\{x \in G : c(x) \neq c(g^{-1}xh)\}\right| \leq \lambda\}$: The union of the V_λ being obviously $L(G)$, we can build a chain C of subgroups K of $L(G)$ of cardinality $< |G|$ with the following properties:

(a) Every K of C is included in some V_λ.
(b) The union of the K is $L(G)$.

Now let us prove that for every element K of C,
$\left|\{x \in G : \exists (u, v) \in K (x^2 = uv)\}\right| < |G|$.

Assume, on the contrary, that there is K of C such that
$\left|\{x \in G : \exists (u, v) \in K (x^2 = uv)\}\right| = |G|$.

For this K, we denote by U the subset of G defined as $U = \{x \in G : \exists (u, v) \in K\ (x^2 = uv)\}$. We have $|U| = |G|$. We denote by S the subset of G defined as $S = \{x \in G : \exists (u, v) \in K, c(x) \neq c(u^{-1}xv)\}$. Since the cardinality of K is $< |G|$ and since K is included in a V_λ, $|S| < |G|$. Thus let M be the subset of G defined as $M = \{x \in G : \forall(u, v) \in K\ (c(x) = c(u^{-1}xv))$ and $\exists(u, v) \in K\ ((ux^{-1})^2 = uv)\}$. Since $U \backslash S$ is included in M, we have $|M| = |G|$.

But $M = \{x \in G : \forall(u, v) \in K\ (c(x) = c(u^{-1}xv))$ and $\exists(u, v) \in K\ (x^{-1} = u^{-1}xv)\}$.

Let $A = \{x \in G : c(x) = c(x^{-1})\}$. The set A includes M, thus $|A| = |G|$. We have a subset A of G of cardinality $|G|$ such that A is symmetric with respect to 1. Thus G includes a monochromatic subset of cardinality $|G|$. This is a contradiction.

(2) $\Rightarrow$ (1): Let C be a chain with the properties of (2). Without loss of generality we can suppose that for every K of C, if $(x, y) \in K$ then $(y, x) \in K$ (symmetry property). □

Definition. Let K be a subgroup of $L(G)$. We will say that two elements x and y of G are *K-equivalent* if there exists (g, h) element of $K \subset L(G)$ such that $y = g^{-1}xh$.

The K-equivalence so defined is obviously an equivalence relation.

Let x be an element of G. Denote by $K(x)$ the union of the elements K of C such that x is not K-equivalent to x^{-1} if this union is not empty.

For each subgroup K of G such that if $(x, y) \in K$ then $(y, x) \in K$, consider a color function $c(K)$ from G to $\{0, 1\}$ such that

(1) If x is K-equivalent to y, then $c(K)(x) = c(K)(y)$
 ($c(K)$ induces a coloring on the K-classes).
(2) If x is not K-equivalent to x^{-1}, then $c(K)(x) \neq c(K)(x^{-1})$
 (the symmetry property of K implies that if H is a K-class, H^{-1} is also a K-class).

According to the Axiom of Choice, such a $c(K)$ exists.

Now we define the following "color function" c from G to $\{0, 1\}$:

Let x be an element of G. If for every K of C, x is K-equivalent to x^{-1}, then $c(x) = 0$.

Otherwise $c(x) = c(K(x))(x)$. Let g be an element of G. There is an element K_g of C such that $(g^{-1}, g) \in K_g$. So, for every element x of G, $gx^{-1}g$ is K_g-equivalent to x^{-1}.

Let $N = \{x \in G : x$ is K_g-equivalent to $x^{-1}\}$. Let x be an element of N. There is $(u, v) \in K_g$ such that $x^{-1} = u^{-1}xv$ which is equivalent to say that $(ux^{-1})^2 = uv$.

Since $\left|\{x \in G : \exists(u, v) \in K_g(x^2 = uv)\}\right| < |G|$ (because the chain C satisfies (2)) and $\left|K_g\right| < |G|$, we can conclude that $|N| < |G|$.

But by the definition of N, for all $x \in G \backslash N$, we have $K_g \subset K(x)$. As $gx^{-1}g$ is K_g-equivalent to x^{-1}, it follows that $K(x) = K(gx^{-1}g)$. By construction of c, for every x of $G \backslash N$, $c(x) \neq c(gx^{-1}g)$. Thus we have $\{x \in G : c(x) = c(gx^{-1}g)\} \subset N$; further as we mentioned before, the latter set has cardinality $< |G|$. It follows that there is no symmetric monochromatic subset of G of cardinality $|G|$. This concludes the proof.

In Theorem 1, we have proved that a group admitting a 2-coloring without infinite monochrome subsets is either locally finite countable or almost cyclic. In Theorem 4, we have given a necessary and sufficient condition for an uncountable or countable locally finite group G to admit a 2-coloring without monochromatic symmetric subsets of cardinality $|G|$. To have a necessary and sufficient condition for an arbitrary group to admit a 2-coloring without infinite monochromatic subsets, it suffices to study the almost cyclic case. This will be done in Theorem 5.

Theorem 5. *Let G be an almost cyclic group, let H be a normal cyclic subgroup of finite index of G.*

Let $C(H)$ be the centralizer of H in G. Let F be the subgroup of G generated by $[G, G]$ and the squares of $C(H)$. G admits a 2-coloring without infinite monochromatic symmetric subsets if and only if either $C(H) = G$ or there is a square in G which is not in F.

Proof. Since H is a normal cyclic subgroup of finite index of an almost cyclic group G, H is isomorphic to $(\mathbb{Z}, +)$. It follows that the only possible conjugates of a are a and a^{-1}. Thus either $C(H) = G$ or $C(H)$ is a subgroup of G of index 2.

Assume that G admits a 2-coloring c without infinite monochromatic symmetric subsets. We will prove that either $C(H) = G$ or there is a square in G which is not in F. Let a be a generator of H. According to Lemma 1 of Theorem 1, for every x of $C(H)$, for every d of $[G, G]$ and for every h of H, we have $c(dx^2h) = c(dxhx) = c(h)$ except at most for finitely many h of H. Thus for every f of F and every h of H, we have $c(fh) = c(h)$ except at most for a finite number of h of H. If $C(H) \neq G$, let u be an element of $G \backslash C(H)$. For every h of H, we have $uhu = u^2h^{-1}$. According to Lemma 1 of Theorem 1, $c(uhu) = c(u)$ except for a finite number of h of H. But $c(h) \neq c(h^{-1})$ except for finitely many h of H. It follows that $c(u^2h^{-1}) \neq c(h^{-1})$ except for a finite number of h of H. We can conclude that $u^2 \notin F$.

Conversely, we consider first the case where $c(H) = G$.

Let a be a generator of H. Let J be a minimal subset of G such that $JH = \{xh : x \in J$ and $h \in H\} = G$.

We define $c(xh) = 0$ ($x \in J$ and $h \in H$) if h is a nonnegative power of a and $c(xh) = 1$ if h is a negative power of a. Let d be a fixed element of G. Let f be an element of G, f is of the form xa^r ($x \in J$ and $r \in \mathbb{Z}$). Since $C(H) = G$, a commutes with every element of G. We obtain $df^{-1}d = dx^{-1}da^{-r}$. But there exists an unique couple (x', s) of $J \times \mathbb{Z}$ such that $dx^{-1}d = x'a^s$. Then we have $df^{-1}d = x'a^{s-r}$. We conclude that $c(f) = c(df^{-1}d)$ if and only if either r and $s - r$ are both nonnegative or there are both negative. For fixed d and x, this is possible with only finitely many r of $\mathbb{Z}$. Since J is finite, we deduce that $c(g) \neq c(dg^{-1}d)$ except for a finite number of g of G. This means that there is no infinite monochromatic subsets of G which is symmetric with respect to d.

We now consider the case where there is a square in G which is not in F.

Since the only possible conjugates of a are a and a^{-1}, the index of $C(H)$ in G is at most 2. It follows that $C(H)$ includes F. By construction, G/F is a finite abelian 2-group and $C(H)/F$ can be identified to a $\mathbb{Z}/2\mathbb{Z}$ vector space.

If there is a square of G which is not in F, the image of this square by $G \to G/F$ is a nontrivial element of $C(H)/F$. Denote this element by w. If we take a subspace of codimension 1 of $C(H)/F$ which does not contain w, we obtain a normal subgroup P of G of index 4 including F and included in $C(H)$ such that P does not contain this square. It follows that G/P is isomorphic to $(\mathbb{Z}/4\mathbb{Z}, +)$.

Let a be a generator of $H \cap P$. Let J be a minimal subset of P such that $J(H \cap P) = \{xh : x \in J$ and $h \in H \cap P\} = P$. We define c on P in the same way as on G in the case where $C(H) = G$ with

$H \cap P$ at the place of H.

Let u be a fixed element of $G \backslash C(H)$, the image of u by $G \to G/P$ is a generator of G/P. For every element p of P, we define $c(up) = c(p)$, $c(u^2 p) \neq c(p)$, $c(u^3 p) \neq c(p)$. We have defined a 2-coloring on G without infinite symmetric monochromatic subsets.

In the following lines, we give a verification.

Let a be a generator of $H \cap P$. Let d be a fixed element of G, d is of the form $u^i q$ ($i \in \{0, 1, 2, 3\}$, $q \in P$). Let f be an element of G, f is of the form $u^j x a^r$ ($j \in \{0, 1, 2, 3\}$, $x \in J$ and $r \in \mathbb{Z}$). Since a belongs to H and since x, q, u^2 belong to $C(H)$, a commutes with x, q and u^2.

We consider first the case where $i - j$ is even.

Then a commutes with u^{i-j}. We obtain $df^{-1}d = u^i q x^{-1} u^{i-j} q a^{-r}$.

We have $2i - j \equiv j$ modulo 4. It follows that we have an unique couple (x', s) of $J \times \mathbb{Z}$ such that $u^i q x^{-1} u^{i-j} q = u^j x' a^s$.

Then if $c(f) = c(df^{-1}d)$, we obtain $c(u^j x a^r) = c(u^j x' a^{s-r})$ which implies that $c(xa^r) = c(x' a^{s-r})$.

We conclude that if $c(f) = c(df^{-1}d)$, either r and $s - r$ are both nonnegative or they are both negative. For fixed d, x and j, this is possible with only finitely many r of $\mathbb{Z}$.

We now consider the case where $i - j$ is odd.

The element u^{j-i} is not in $C(H)$. We know that H is a normal subgroup of G and that H is isomorphic to $(\mathbb{Z}, +)$. We conclude that for every element h of H, $u^{j-i} h u^{i-j} = h^{-1}$.

Then $u^{j-i} a u^{i-j} = a^{-1}$.

We obtain $df^{-1}d = u^i q x^{-1} u^{i-j} q a^r$.

We have $2i - j \equiv j + 2$ modulo 4. If $k \in \{0, 1, 2, 3\}$ and $k \equiv j + 2$ modulo 4, we conclude that we have an unique couple (x', s) of $J \times \mathbb{Z}$ such that $u^i q x^{-1} u^{i-j} q = u^k x' a^s$.

Then if $c(f) = c(df^{-1}d)$, we obtain $c(u^j x a^r) = c(u^k x' a^{s+r})$.

The only possible couples (j, k) are (0, 2), (2, 0), (1, 3) and (3, 1).

It follows that $c(u^j x a^r) = c(u^k x' a^{s+r})$ if and only if $c(xa^r) \neq c(x' a^{s+r})$.

We conclude that if $c(f) = c(df^{-1}d)$ if and only if either r is negative and $s + r$ is nonnegative or r is nonnegative and $s + r$ is negative. For fixed d, x and j ; this is possible with only finitely many r of $\mathbb{Z}$.

Since J and $\{0, 1, 2, 3\}$ are finite, we deduce from these 2 cases that $c(g) \neq c(dg^{-1}d)$ except for a finite number of g of G. This means that there is no infinite monochromatic subsets of G which is symmetric with respect to d. □

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